

# REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH REEB PARALLEL RICCI TENSOR IN GENERALIZED TANAKA-WEBSTER CONNECTION

HYUNJIN LEE, YOUNG JIN SUH AND CHANGHWA WOO

**ABSTRACT.** There are several kinds of classification problems for real hypersurfaces in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ . Among them, Suh classified Hopf hypersurfaces  $M$  in  $G_2(\mathbb{C}^{m+2})$  with Reeb parallel Ricci tensor in Levi-Civita connection. In this paper, we introduce a new notion of generalized Tanaka-Webster Reeb parallel Ricci tensor for  $M$  in  $G_2(\mathbb{C}^{m+2})$ . By using such parallel conditions, we give complete classifications of Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$ .

## INTRODUCTION

In this paper, let  $M$  represent a real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , and  $S$  denote the Ricci tensor of  $M$ . Hereafter unless otherwise stated, we consider that  $X, Y$ , and  $Z$  are any tangent vector fields on  $M$ . Let  $W$  be any tangent vector field on the distribution  $\mathfrak{h} = \{X \in TM \mid X \perp \xi\}$ .  $k$  stands for a non-zero constant real number.

The classification of real hypersurfaces in Hermitian symmetric space is one of interesting parts in the field of differential geometry. Among them, we introduce a complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2})$  defined by the set of all complex two-dimensional linear subspaces in  $\mathbb{C}^{m+2}$ . It is a kind of Hermitian symmetric space of compact irreducible type with rank 2. Remarkably, the manifolds are equipped with both a Kähler structure  $J$  and a quaternionic Kähler structure  $\mathfrak{J}$  satisfying  $JJ_\nu = J_\nu J$  ( $\nu = 1, 2, 3$ ) where  $\{J_\nu\}_{\nu=1,2,3}$  is an orthonormal basis of  $\mathfrak{J}$ . When  $m = 1$ ,  $G_2(\mathbb{C}^3)$  is isometric to the two-dimensional complex projective space  $\mathbb{C}P^2$  with constant holomorphic sectional curvature eight. When  $m = 2$ , we note that the isomorphism  $\text{Spin}(6) \simeq \text{SU}(4)$  yields an isometry between  $G_2(\mathbb{C}^4)$  and the real Grassmann Manifold  $G_2^+(\mathbb{R}^6)$  of oriented two-dimensional linear subspaces in  $\mathbb{R}^6$ . In this paper we assume  $m$  is not less than 3. (see [2]).

Let  $N$  be a local unit normal vector field of  $M$ . Since  $G_2(\mathbb{C}^{m+2})$  has the Kähler structure  $J$ , we may define a *Reeb vector field*  $\xi = -JN$  and a 1-dimensional distribution  $[\xi] = \text{Span}\{\xi\}$ . The Reeb vector field  $\xi$  is said to be a *Hopf* if it is

<sup>1</sup>2010 Mathematics Subject Classification : Primary 53C40; Secondary 53C15.

<sup>2</sup>Key words : Real hypersurfaces; complex two-plane Grassmannians; Hopf hypersurface; generalized Tanaka-Webster connection; Ricci tensor; Reeb parallel.

\* This work was supported by Grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea. The first author by Grant Proj. No. NRF-2012-R1A1A3002031, the second by Grant Proj. No. NRF-2012-R1A2A2A01043023. And the third author supported by NRF Grant funded by the Korean Government (NRF-2013-Fostering Core Leaders of Future Basic Science Program).

invariant under the shape operator  $A$  of  $M$ . The 1-dimensional foliation of  $M$  by the integral curves of  $\xi$  is said to be a *Hopf foliation* of  $M$ . We say that  $M$  is a *Hopf hypersurface* if and if the Hopf foliation of  $M$  is totally geodesic. By the formulas in [9, Section 2], it can be easily seen that  $\xi$  is Hopf if and only if  $M$  is Hopf.

From the quaternionic Kähler structure  $\mathfrak{J}$  of  $G_2(\mathbb{C}^{m+2})$ , there naturally exists *almost contact 3-structure* vector field  $\xi_\nu = -J_\nu N$ ,  $\nu = 1, 2, 3$ . Put  $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ , which is a 3-dimensional distribution in a tangent vector space  $T_x M$  of  $M$  at  $x \in M$ . In addition,  $\mathcal{Q}$  stands for the orthogonal complement of  $\mathcal{Q}^\perp$  in  $T_x M$ . It becomes the quaternionic maximal subbundle of  $T_x M$ . Thus the tangent space of  $M$  consists of the direct sum of  $\mathcal{Q}$  and  $\mathcal{Q}^\perp$  as follows:  $T_x M = \mathcal{Q} \oplus \mathcal{Q}^\perp$ .

For two distributions  $[\xi]$  and  $\mathcal{Q}^\perp$  defined above, we may consider two natural invariant geometric properties under the shape operator  $A$  of  $M$ , that is,  $A[\xi] \subset [\xi]$  and  $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$ . By using the result of Alekseevskii [1], Berndt and Suh [2] have classified all real hypersurfaces with two natural invariant properties in  $G_2(\mathbb{C}^{m+2})$  as follows:

**Theorem A.** *Let  $M$  be a real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . Then both  $[\xi]$  and  $\mathcal{Q}^\perp$  are invariant under the shape operator of  $M$  if and only if*

- (A)  *$M$  is an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ , or*
- (B)  *$m$  is even, say  $m = 2n$ , and  $M$  is an open part of a tube around a totally geodesic  $\mathbb{H}P^n$  in  $G_2(\mathbb{C}^{m+2})$ .*

In the case (A), we say  $M$  is of Type (A). Similarly in the case (B) we say  $M$  is of Type (B). Using Theorem A, geomtricians have given characterizations for Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  with geometric quantities, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. Actually, Lee and Suh [9] gave a characterization for a real hypersurface of Type (B) as follows:

**Theorem B.** *Let  $M$  be a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . Then  $\xi$  belongs to the distribution  $\mathcal{Q}$  if and only if  $M$  is locally congruent to an open part of a tube around a totally geodesic  $\mathbb{H}P^n$  in  $G_2(\mathbb{C}^{m+2})$ ,  $m = 2n$ . In other words,  $M$  is locally congruent to a real hypersurface of Type (B).*

In particular, there are various well-known results with respect to  $S$  on Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$ . From such a point of view, Suh [17] gave a characterization of a model space of Type (A) in  $G_2(\mathbb{C}^{m+2})$  under the condition  $S\phi = \phi S$  where  $\phi$  denotes the structure tensor field of  $M$ . In [18] and [19], he also considered the parallelism of Ricci tensor with respect to the Levi-Civita connection and gave, respectively,

**Theorem C.** [19] *Let  $M$  be a real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$  with non-vanishing geodesic Reeb flow. If the Ricci tensor is Reeb parallel,  $\nabla_\xi S = 0$ . Then  $M$  is locally congruent to one of the following:*

- (i) *a tube over a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$  with radius  $r \neq \frac{\pi}{4\sqrt{2}}$ , or*
- (ii) *a tube over a totally geodesic  $\mathbb{H}P^n$ ,  $m = 2n$ , in  $G_2(\mathbb{C}^{m+2})$  with radius  $r$  such that  $\cot^2(2r) = \frac{1}{2m-1}$  and  $\xi$ -parallel eigenspaces  $T_{\cot r}$  and  $T_{\tan r}$ .*

Motivated by these works, we define the notion of Reeb parallel Ricci tensor with respect to the generalized Tanaka-Webster connection for a real hypersurface  $M$

in  $G_2(\mathbb{C}^{m+2})$ . In order to do this, we first define the generalized Tanaka-Webster connection  $\widehat{\nabla}^{(k)}$  on  $M$  given by

$$\widehat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,$$

where  $k$  is a non-zero real number (see [3], [4], [5]). Hereafter, unless otherwise stated, a GTW connection means a generalized Tanaka-Webster connection. In addition, we put

$$F_X^{(k)} Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$

Then the operator  $F_X^{(k)}$  becomes a skew-symmetric (1,1) type tensor, that is,  $g(F_X^{(k)} Y, Z) = -g(Y, F_X^{(k)} Z)$  for any tangent vector fields  $X, Y$ , and  $Z$  on  $M$  and said to be *Tanaka-Webster* (or *k-th-Cho*) *operator* with respect to  $X$ .

Related to this connection, the Ricci tensor  $S$  is said to be *generalized Tanaka-Webster Reeb parallel* (in short, *GTW-Reeb parallel*) if the covariant derivative in GTW connection  $\widehat{\nabla}^{(k)}$  of  $S$  along  $\xi$  is vanishing, that is,  $(\widehat{\nabla}_\xi^{(k)} S)Y = 0$ . From this, we naturally see that this notion is weaker than generalized Tanaka-Webster parallel (shortly, GTW-parallel) Ricci tensor, that is,  $(\widehat{\nabla}_X^{(k)} S)Y = 0$ . Recently, Pérez and Suh [14] proved the non-existence of Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with GTW-parallel Ricci tensor. From such a viewpoint, we assert:

**Theorem 1.** *Let  $M$  be a Hopf hypersurface in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with  $\alpha = g(A\xi, \xi) \neq 2k$ . The Ricci tensor  $S$  of  $M$  is GTW-Reeb parallel if and only if  $M$  is locally congruent to one of the following:*

- (i) *a tube over a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$  with radius  $r$  such that  $r \neq \frac{1}{2\sqrt{2}} \cot^{-1}(\frac{k}{\sqrt{2}})$ , or*
- (ii) *a tube over a totally geodesic  $\mathbb{H}P^n$ ,  $m = 2n$ , in  $G_2(\mathbb{C}^{m+2})$  with radius  $r$  such that  $r = \frac{1}{2} \cot^{-1}(\frac{-k}{4(2n-1)})$ .*

For the case  $\alpha = 2k$ , the Reeb vector field  $\xi$  of Hopf hypersurface  $M$  with GTW-Reeb parallel Ricci tensor belongs to either  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$ . So, for the case  $\xi \in \mathcal{Q}^\perp$ , we obtain that the trace  $h$  of the shape operator  $A$  is constant along  $\xi$ , that is,  $\xi h = 0$ . In addition for the case  $\xi \in \mathcal{Q}$  we have the following:

**Corollary 1.** *Let  $M$  be a real hypersurface in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with GTW-Reeb parallel Ricci tensor for  $\alpha = 2k$ . If  $\xi$  belongs to the distribution  $\mathcal{Q}$ , then  $M$  is locally congruent to an open part of a tube around a totally geodesic  $\mathbb{H}P^n$ ,  $m = 2n$ , in  $G_2(\mathbb{C}^{m+2})$  with radius  $r$  such that  $r = \frac{1}{2} \tan^{-1} \sqrt{2n-1}$ .*

On the other hand, we consider the notion of GTW-Reeb parallel Ricci tensor on  $\mathfrak{h}$ , that is,  $(\widehat{\nabla}_\xi^{(k)} S)W = 0$  for any  $W \in \mathfrak{h}$ . Then by virtue of Theorem C for the case  $\alpha = 2k$ , we assert the following:

**Theorem 2.** *Let  $M$  be a Hopf hypersurface in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with  $\alpha = 2k$ . The Ricci tensor of  $M$  satisfies the Reeb parallelism on  $\mathfrak{h}$  in both GTW and Levi-Civita connections, that is,  $(\widehat{\nabla}_\xi^{(k)} S)W = 0$  and  $(\nabla_\xi S)W = 0$  for any  $W \in \mathfrak{h}$  if and only if  $M$  is locally congruent to an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$  with radius  $r$  such that  $r = \frac{1}{2\sqrt{2}} \cot^{-1}(\frac{k}{\sqrt{2}})$ .*

Moreover, as a generalization of the assumption  $\widehat{\nabla}_\xi^{(k)} S = 0 = \nabla_\xi S$  on  $\mathfrak{h}$  in Theorem 2, we want to consider that  $\widehat{\nabla}_\xi^{(k)} S = \nabla_\xi S$ , that is, the Reeb parallel Ricci tensor in GTW connection coincides with the Reeb parallel Ricci tensor in Levi-Civita connection. This condition has a geometric meaning such that  $S$  commutes with the Tanaka-Webster operator  $F_\xi$ , that is,  $S \cdot F_\xi = F_\xi \cdot S$ . This meaning gives any eigenspaces of  $S$  are invariant by the Tanaka-Webster operator  $F_\xi$ . From such a point of a view, we have the following:

**Theorem 3.** *Let  $M$  be a Hopf hypersurface in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . Then  $\widehat{\nabla}_\xi^{(k)} S = \nabla_\xi S$  if and only if  $M$  is locally congruent to an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ .*

But for the case where the derivative of the Ricci tensor in GTW connection is equal to the derivative in Levi-Civita connection, that is,  $\widehat{\nabla}_X^{(k)} S = \nabla_X S$  for any  $X \in TM$ , we assert the following:

**Corollary 2.** *There does not exist any Hopf hypersurface in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , satisfying  $(\widehat{\nabla}_X^{(k)} S)Y = (\nabla_X S)Y$  for arbitrary tangent vector fields  $X$  and  $Y$  on  $M$ .*

Obviously, we know that the condition  $\widehat{\nabla}_X^{(k)} S = \nabla_X S$  has a geometric meaning that any eigenspaces of  $S$  are invariant by the Tanaka-Webster operator  $F_X$ . Recently, Pérez and Suh [15] investigated the Levi-Civita and GTW covariant derivatives for the shape operator or the structure Jacobi operator of real hypersurfaces in complex projective space  $\mathbb{C}P^m$ . Moreover, in [6] Jeong, Lee and Suh gave a characterization of Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  with  $\widehat{\nabla}^{(k)} A = \nabla A$ .

In this paper, we refer [1], [2], [7], [9], [16] and [17] for Riemannian geometric structures of  $G_2(\mathbb{C}^{m+2})$  and its geometric quantities, respectively. In order to get our results, in sections 1 we will give the fundamental formulas related to the Reeb parallel Ricci tensor. In section 2, we want to give a complete proof of Theorem 1 for  $\alpha = g(A\xi, \xi) \neq 2k$ . In section 3 we will consider the case  $\alpha = 2k$  and give a proof of Corollary 1 and Theorem 2. Finally, in section 4 we will give a complete proof of Theorem 3 and Corollary 2.

## 1. GTW-REEB PARALLEL RICCI TENSOR

From [13], the Ricci tensor  $S$  of a real hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , is given by

$$\begin{aligned}
 SX &= \sum_{i=1}^{4m-1} R(X, e_i)e_i \\
 (3.1) \quad &= (4m+7)X - 3\eta(X)\xi + hAX - A^2X \\
 &\quad + \sum_{\nu=1}^3 \{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\}
 \end{aligned}$$

where  $h$  denotes the trace of the shape operator  $A$ , that is,  $h = \text{Tr} A$ .

And we also have

$$\begin{aligned}
 (\nabla_X S)Y &= -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\
 &\quad - 3 \sum_{\nu=1}^3 \left\{ g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \right\} \\
 (3.2) \quad &\quad + \sum_{\nu=1}^3 \left\{ 2g(\phi AX, \xi_\nu)\phi_\nu \phi Y + g(AX, \phi_\nu \phi Y)\phi_\nu \xi \right. \\
 &\quad \left. - \eta(Y)g(AX, \xi_\nu)\phi_\nu \xi + \eta_\nu(\phi Y)g(AX, \xi)\xi_\nu - \eta_\nu(\phi Y)\phi_\nu \phi AX \right. \\
 &\quad \left. - \eta(Y)g(\phi AX, \xi_\nu)\xi_\nu - \eta(Y)g(\phi_\nu AX, \xi)\xi_\nu \right\} \\
 &\quad + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.
 \end{aligned}$$

Substituting  $X = \xi$  into (3.2) and using the condition that  $M$  is Hopf, that is,  $A\xi = \alpha\xi$ , we get

$$\begin{aligned}
 (3.3) \quad (\nabla_\xi S)Y &= -4\alpha \sum_{\nu=1}^3 \left\{ g(\phi_\nu \xi, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu \xi \right\} + (\xi h)AY \\
 &\quad + h(\nabla_\xi A)Y - (\nabla_\xi A)AY - A(\nabla_\xi A)Y.
 \end{aligned}$$

In this section we assume that  $M$  is a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$  with GTW-Reeb parallel Ricci tensor, that is,  $S$  satisfies:

$$(C-1) \quad (\hat{\nabla}_\xi^{(k)} S)X = 0.$$

By the definition of GTW connection  $\hat{\nabla}^{(k)}$ , the covariant derivative of  $S$  with respect to the GTW connection along  $\xi$  becomes

$$\begin{aligned}
 (3.4) \quad (\hat{\nabla}_\xi^{(k)} S)X &= \hat{\nabla}_\xi^{(k)}(SX) - S(\hat{\nabla}_\xi^{(k)} X) \\
 &= \nabla_\xi(SX) + g(\phi A\xi, SX)\xi - \eta(SX)\phi A\xi - k\eta(\xi)\phi SX \\
 &\quad - S(\nabla_\xi X) - g(\phi A\xi, X)S\xi + \eta(X)S\phi A\xi + k\eta(\xi)S\phi X \\
 &= (\nabla_\xi S)X - k\phi SX + kS\phi X.
 \end{aligned}$$

Thus the condition (C-1) is equivalent to

$$(3.5) \quad (\nabla_\xi S)X = k\phi SX - kS\phi X,$$

it yields

$$\begin{aligned}
 (3.6) \quad &4(k - \alpha) \sum_{\nu=1}^3 \left\{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu \xi \right\} \\
 &= (\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X - kh\phi AX \\
 &\quad + k\phi A^2 X + khA\phi X - kA^2 \phi X
 \end{aligned}$$

from (3.1), (3.2) and [8, Section 2].

Using these equations, we prove that  $\xi$  belongs to either  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$ , where  $M$  is a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$  with GTW-Reeb parallel Ricci tensor.

**Lemma 1.1.** *Let  $M$  be a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . If  $M$  has GTW-Reeb parallel Ricci tensor, then  $\xi$  belongs to either  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$ .*

*Proof.* In order to prove this lemma, we put

$$(**) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vectors  $X_0 \in \mathcal{Q}$  and  $\xi_1 \in \mathcal{Q}^\perp$ . Putting  $X = \xi$  in (3.6), by (\*\*) and basic formulas in [8, Section 2], it follows that

$$(3.7) \quad 4(k - \alpha)\eta_1(\xi)\phi_1\xi = \alpha(\xi h)\xi - h(\xi\alpha)\xi - 2\alpha(\xi\alpha)\xi,$$

where we have used  $(\nabla_\xi A)\xi = (\xi\alpha)\xi$  and  $(\nabla_\xi A)A\xi = \alpha(\xi\alpha)\xi$ .

Taking the inner product of (3.7) with  $\phi_1\xi$ , we have

$$(3.8) \quad 4(k - \alpha)\eta_1(\xi)\eta^2(X_0) = 0,$$

because of  $\eta^2(X_0) + \eta^2(\xi_1) = 1$ . From this, we have the following three cases.

**Case 1:**  $\alpha = k$ .

For this case, we see that  $\alpha$  becomes a non-zero real number. Using the equation in [2, Lemma 1], we assert that  $\xi$  belongs to either  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$ .

**Case 2:**  $\eta(\xi_1) = 0$ .

By the notation (\*\*), we see that  $\xi$  belongs to  $\mathcal{Q}$ .

**Case 3:**  $\eta(X_0) = 0$ .

This case implies that  $\xi$  belongs to  $\mathcal{Q}^\perp$  from (\*\*).

Accordingly, summing up these cases, the proof of our Lemma is completed.  $\square$

## 2. PROOF OF THEOREM 1

In this section, let  $M$  be a Hopf hypersurface,  $\alpha \neq 2k$ , in  $G_2(\mathbb{C}^{m+2})$  with GTW-Reeb parallel Ricci tensor. Then by Lemma 1.1 we shall divide our consideration in two cases depending on  $\xi$  belongs to either  $\mathcal{Q}^\perp$  or  $\mathcal{Q}$ , respectively.

First of all, if we assume  $\xi \in \mathcal{Q}$ , then a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with GTW-Reeb parallel Ricci tensor and  $\alpha = g(A\xi, \xi) \neq 2k$  is locally congruent to a real hypersurface of Type (B) by virtue of Theorem B given in the introduction.

Next let us consider the case,  $\xi \in \mathcal{Q}^\perp$ . Accordingly, we may put  $\xi = \xi_1$ . Since  $M$  is a Hopf hypersurface with GTW-Reeb parallel Ricci tensor, the equation (3.6) becomes

$$(4.1) \quad \begin{aligned} & (\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X \\ & = k(h\phi AX - \phi A^2X - hA\phi X + A^2\phi X). \end{aligned}$$

From the Codazzi equation [8, Section 2] and differentiating  $A\xi = \alpha\xi$ , we obtain

$$\begin{aligned} (\nabla_\xi A)X &= (\nabla_X A)\xi + \phi X + \phi_1X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \\ &= (X\alpha)\xi + \alpha\phi AX - A\phi AX + \phi X + \phi_1X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \end{aligned}$$

Using the equation [8, Lemma 2.1] and the previous one, we get

$$(\nabla_\xi A)X = \frac{\alpha}{2}\phi AX - \frac{\alpha}{2}A\phi X + (\xi\alpha)\eta(X)\xi.$$

Therefore from this, (4.1) can be written as

$$(4.2) \quad (\xi h)AX + \tilde{\kappa}h\phi AX - \tilde{\kappa}hA\phi X + (h - 2\alpha)(\xi\alpha)\eta(X)\xi - \tilde{\kappa}\phi A^2X + \tilde{\kappa}A^2\phi X = 0,$$

where  $\tilde{\kappa} = (\frac{\alpha}{2} - k)$ .

Since  $\tilde{\kappa} \neq 0$  is equivalent to the given condition  $\alpha \neq 2k$ , (4.2) yields

$$(4.3) \quad \frac{(\xi h)}{\tilde{\kappa}} AX + h\phi AX - hA\phi X + \frac{(h-2\alpha)}{\tilde{\kappa}} (\xi\alpha)\eta(X)\xi - \phi A^2 X + A^2\phi X = 0.$$

Now we consider the case  $\xi h = 0$ . Then (4.3) can be reduced to

$$(4.4) \quad h\phi AX - hA\phi X + \frac{(h-2\alpha)}{\tilde{\kappa}} (\xi\alpha)\eta(X)\xi - \phi A^2 X + A^2\phi X = 0.$$

Taking the inner product of (4.4) with  $\xi$ , we have  $\frac{(h-2\alpha)}{\tilde{\kappa}} (\xi\alpha)\eta(X) = 0$ . Thus (4.4) becomes

$$(4.5) \quad h\phi AX - \phi A^2 X - hA\phi X + A^2\phi X = 0.$$

On the other hand, from the equation (3.1) we calculate

$$S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X,$$

then by (4.5) it follows that  $S\phi X = \phi SX$  for any tangent vector field  $X$  on  $M$ . Hence, by Suh [17] we assert that  $M$  satisfying our assumptions must be a model space of Type (A).

We now assume  $\xi h \neq 0$ . Putting  $\sigma = \frac{(\xi h)}{\tilde{\kappa}} (\neq 0)$  and  $\tau = \frac{(h-2\alpha)}{\tilde{\kappa}} (\xi\alpha)$ , the equation (4.3) becomes

$$(4.6) \quad \sigma AX + h\phi AX - hA\phi X + \tau\eta(X)\xi - \phi A^2 X + A^2\phi X = 0.$$

Applying  $\phi$  to (4.6) and replacing  $X$  by  $\phi X$  in (4.6), respectively, we get the following two equations:

$$\sigma\phi AX - hAX + h\alpha\eta(X)\xi - h\phi A\phi X + A^2 X - \alpha^2\eta(X)\xi + \phi A^2\phi X = 0$$

and

$$\sigma A\phi X + h\phi A\phi X + hAX - h\alpha\eta(X)\xi - \phi A^2\phi X - A^2 X + \alpha^2\eta(X)\xi = 0.$$

Summing up the above two equations, we obtain  $\phi A + A\phi = 0$ . Thus from this, the equation (4.6) implies

$$\sigma AX + 2h\phi AX + \tau\eta(X)\xi = 0.$$

Let us  $X_{\mathfrak{h}}$  be the orthogonal projection of  $X$  onto the distribution  $\mathfrak{h} = \{X \in TM \mid X \perp \xi\}$ . Inserting this into the previous equation yields

$$\sigma AX_{\mathfrak{h}} + 2h\phi AX_{\mathfrak{h}} = 0.$$

In addition, applying  $\phi$  to this equation, it follows

$$\sigma\phi AX_{\mathfrak{h}} - 2hAX_{\mathfrak{h}} = 0.$$

Thus we obtain

$$\begin{pmatrix} \sigma & 2h \\ -2h & \sigma \end{pmatrix} \begin{pmatrix} AX_{\mathfrak{h}} \\ \phi AX_{\mathfrak{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the square matrix of order 2, that is,  $\sigma^2 + 4h^2 \geq \sigma^2 \neq 0$ , so we get  $AX_{\mathfrak{h}} = 0$  for any  $X_{\mathfrak{h}} \in \mathfrak{h}$ . Substituting  $X_{\mathfrak{h}}$  as  $\xi_2$  and  $\xi_3$ , it implies  $A\xi_2 = 0$  and  $A\xi_3 = 0$ , respectively. Hence, we can assert that the distribution  $\mathcal{Q}^\perp$  is invariant under the shape operator, that is,  $M$  is a  $\mathcal{Q}^\perp$ -invariant real hypersurface. Thus by virtue of Theorem A, we conclude that  $M$  with our assumptions must be a model space of Type (A).

Summing up these discussions, we conclude that if a Hopf hypersurface  $M$  in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , satisfying (C-1) and  $\alpha \neq 2k$ , then  $M$  is of Type (A) or (B).

Hereafter, let us check whether  $S$  of a model space of Type (A) (or of Type (B)) satisfies the Reeb parallelism with respect to  $\widehat{\nabla}^{(k)}$  by [2, Proposition 3] (or [2, Proposition 2], respectively).

Let us denote by  $M_A$  a model space of Type (A). From now on, using the equations (3.1), (3.2) and [2, Proposition 3], let us check whether or not  $S$  satisfies (3.6) which is equivalent to our condition (C-1) for each eigenspace  $T_\alpha$ ,  $T_\beta$ ,  $T_\lambda$ , and  $T_\mu$  on  $T_x M_A$ ,  $x \in M_A$ . In order to do, we find one equation related to  $S$  from (3.6) using the property of  $M_A$ ,  $\xi = \xi_1$  as follows.

$$(4.7) \quad \begin{aligned} (\widehat{\nabla}_\xi^{(k)} S)X &= -h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X + kh\phi AX \\ &\quad - k\phi A^2 X - khA\phi X + kA^2\phi X, \end{aligned}$$

since  $h = \alpha + 2\beta + 2(m-2)\lambda$  is a constant.

**Case A-1:**  $X = \xi (= \xi_1) \in T_\alpha$ .

Since  $(\nabla_\xi A)\xi = 0$ , we see that  $(\widehat{\nabla}_\xi^{(k)} S)\xi = 0$  from the equation (4.7). It means that the Ricci tensor  $S$  becomes GTW Reeb parallel on  $T_\alpha$ .

**Case A-2:**  $X \in T_\beta = \text{Span}\{\xi_2, \xi_3\}$ .

For  $\xi_\mu \in T_\beta$ ,  $\mu = 2, 3$  we have

$$\begin{aligned} (\nabla_\xi A)\xi_\mu &= \beta(\nabla_\xi \xi_\mu) - A(\nabla_\xi \xi_\mu) \\ &= \beta q_{\mu+2}(\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\xi)\xi_{\mu+2} + \alpha\beta\phi_\mu\xi \\ &\quad - q_{\mu+2}(\xi)A\xi_{\mu+1} + q_{\mu+1}(\xi)A\xi_{\mu+2} - \alpha A\phi_\mu\xi, \end{aligned}$$

which follows that  $(\nabla_\xi A)\xi_2 = 0$  and  $(\nabla_\xi A)\xi_3 = 0$ . Therefore, from the equation (4.7) we obtain, respectively,

$$\begin{aligned} (\widehat{\nabla}_\xi^{(k)} S)\xi_2 &= kh\phi A\xi_2 - k\phi A^2\xi_2 - khA\phi\xi_2 + kA^2\phi\xi_2 \\ &= (-kh\beta + k\beta^2 + kh\beta - k\beta^2)\xi_3 = 0, \end{aligned}$$

and  $(\widehat{\nabla}_\xi^{(k)} S)\xi_3 = 0$  by similar methods. So, we assert that the Ricci tensor  $S$  of  $M_A$  is Reeb parallel on  $T_\beta$ .

By the structure of a tangent vector space  $T_x M_A$  at  $x \in M_A$ , we see that the distribution  $\mathcal{Q}$  is composed of two eigenspaces  $T_\lambda$  and  $T_\mu$ . On this distribution  $\mathcal{Q} = T_\lambda \oplus T_\mu$  we obtain

$$(4.8) \quad (\nabla_\xi A)X = \alpha\phi AX - A\phi AX + \phi X + \phi_1 X$$

by virtue of the Codazzi equation [8, Section 2]. Using this equation we consider the following two cases.

**Case A-3:**  $X \in T_\lambda = \{X \mid X \in \mathcal{Q}, JY = J_1 Y\}$ .

We naturally see that if  $X \in T_\lambda$ , then  $\phi X = \phi_1 X$ . Moreover, the vector  $\phi X$  also belong to the eigenspace  $T_\lambda$  for any  $X \in T_\lambda$ , that is,  $\phi T_\lambda \subset T_\lambda$ . From these and (4.8), we obtain

$$(\nabla_\xi A)X = (\alpha\lambda - \lambda^2 + 2)\phi X, \text{ for } X \in T_\lambda.$$



From (4.7) and together with these facts, we obtain

$$(\widehat{\nabla}_\xi^{(k)} S)X = (\alpha\lambda - \lambda^2 + 2)(2\alpha - h)\phi X,$$

which implies that  $S$  must be Reeb parallel for  $\widehat{\nabla}^{(k)}$  on  $T_\lambda$ , since  $(\alpha\lambda - \lambda^2 + 2) = 0$ .

**Case A-4:**  $X \in T_\mu = \{X \mid X \in \mathcal{Q}, JY = -J_1Y\}$ .

If  $X \in T_\mu$ , then  $\phi X = -\phi_1 X$ ,  $\phi T_\mu \subset T_\mu$  and  $\mu = 0$ . So, from (4.8), we obtain  $(\nabla_\xi A)X = 0$ , moreover  $(\widehat{\nabla}_\xi^{(k)} S)X = 0$  for any  $X \in T_\mu$ .

Summing up all cases mentioned above, we can assert that  $S$  of real hypersurfaces  $M_A$  of Type (A) in  $G_2(\mathbb{C}^{m+2})$  is GTW Reeb parallel.

Now let us consider our problem for a model space of Type (B), which will be denoted by  $M_B$ . In order to do this, let us calculate the fundamental equation related to the covariant derivative of  $S$  of  $M_B$  along the direction of  $\xi$  in GTW connection. On  $T_x M_B$ ,  $x \in M_B$ , since  $\xi \in \mathcal{Q}$  and  $h = \text{Tr}(A) = \alpha + (4n - 1)\beta$  is a constant, the equation (3.6) is reduced to

$$\begin{aligned} (\widehat{\nabla}_\xi^{(k)} S)X &= 4(k - \alpha) \sum_{\nu=1}^3 \left\{ \eta_\nu(\phi X) \xi_\nu - \eta_\nu(X) \phi_\nu \xi \right\} \\ &\quad - h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X \\ &\quad + kh\phi AX - k\phi A^2 X - khA\phi X + kA^2 \phi X. \end{aligned}$$

Moreover, by the equation of Codazzi and [2, Proposition 2] we obtain that for any  $X \in T_x M_B$

$$\begin{aligned} (\nabla_\xi A)X &= \alpha\phi AX - A\phi AX + \phi X - \sum_{\nu=1}^3 \left\{ \eta_\nu(X) \phi_\nu \xi + 3g(\phi_\nu \xi, X) \xi_\nu \right\} \\ (4.9) \quad &= \begin{cases} 0 & \text{if } X \in T_\alpha \\ \alpha\beta\phi\xi_\ell & \text{if } X \in T_\beta = \text{Span}\{\xi_\ell \mid \ell = 1, 2, 3\} \\ -4\xi_\ell & \text{if } X \in T_\gamma = \text{Span}\{\phi\xi_\ell \mid \ell = 1, 2, 3\} \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ (\alpha\mu + 2)\phi X & \text{if } X \in T_\mu. \end{cases} \end{aligned}$$

From these two equations, it follows that

$$(4.10) \quad (\widehat{\nabla}_\xi^{(k)} S)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha - k)(4 - h\beta + \beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ (4(\alpha - k) + (h - \beta)(4 + k\beta))\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (h - \beta)(k\lambda - k\mu - \alpha\lambda - 2)\phi X & \text{if } X \in T_\lambda \\ (h - \beta)(k\mu - k\lambda - \alpha\mu - 2)\phi X & \text{if } X \in T_\mu. \end{cases}$$

So, we see that  $M_B$  has Reeb parallel GTW-Ricci tensor, when  $\alpha$  and  $h$  satisfies the conditions  $\alpha = k$  and  $h - \beta = 0$ , which means  $r = \frac{1}{2} \cot^{-1}(\frac{-k}{4(2n-1)})$ . Moreover, this radius  $r$  satisfies our condition  $\alpha \neq 2k$ .

Hence summing up these considerations, we give a complete proof of our Theorem 1 in the introduction.  $\square$

## 3. PROOFS OF COROLLARY 1 AND THEOREM 2

In section 2 we obtained the classification of Hopf hypersurfaces  $M$  with GTW-Reeb parallel Ricci tensor and  $\alpha \neq 2k$ . Thus in present section we will consider the case  $\alpha = 2k$  related to the GTW-Reeb parallelism of Ricci tensor of a Hopf hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ .

Now let us prove Corollary 1 in the introduction.

Our condition  $\alpha = 2k$  means that  $\alpha$  is constant. From this we assert that  $\xi$  belongs to either  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$ . For  $\xi \in \mathcal{Q}$ , it is a well-known fact that a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$  must be a model space  $M_B$  of Type (B) (see [9]). On the other hand, from (4.10) and  $\alpha = 2k$ , the GTW covariant derivative of Ricci tensor  $S$  of  $M_B$  along the direction of  $\xi$  is given

$$(5.1) \quad (\widehat{\nabla}_\xi^{(k)} S)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ k(4 - h\beta + \beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ (4k + (h - \beta)(4 + k\beta))\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ -(h - \beta)(k\beta + 2)\phi X & \text{if } X \in T_\lambda \\ -(h - \beta)(k\beta + 2)\phi X & \text{if } X \in T_\mu. \end{cases}$$

Actually, since  $\alpha = 2k$ , we naturally have  $k\beta + 2 = 0$ . It follows that  $S$  is GTW-Reeb parallel on  $T_\lambda$  and  $T_\mu$ . In order to be the GTW-Reeb parallel Ricci tensor on the other eigenspaces  $T_\beta$  and  $T_\gamma$ , we should have the following two equations,

$$(4 - h\beta + \beta^2) = 0$$

and

$$4k + (h - \beta)(4 + k\beta) = 0.$$

Combining these two equations, we have  $2k + h - \beta = 0$ . Since  $h = \alpha + 3\beta + (4n - 4)(\lambda + \mu) = \alpha + (4n - 1)\beta$  and  $\alpha = 2k$ , it follows that  $\alpha = -(2n - 1)\beta$ . By virtue of [2, Proposition 2],  $\alpha = -2\tan(2r)$  and  $\beta = 2\cot(2r)$  where  $r \in (0, \pi/4)$ , we obtain  $\tan(2r) = \sqrt{2n - 1}$ . From such assertions, we conclude that a model space of Type (B) has GTW-Reeb parallel Ricci tensor for special radius  $r$  such that  $r = \frac{1}{2}\tan^{-1}(\sqrt{2n - 1})$ , which gives us a complete proof of Corollary 1.  $\square$

On the other hand, for the case  $\xi \in \mathcal{Q}^\perp$ , the equation (4.2) becomes

$$(\xi h)AX = 0$$

under the assumption of  $\alpha = 2k$ . For the case  $\xi h \neq 0$ , it follows that  $AX = 0$ . If  $X = \xi$ , then  $\alpha = 0$ , which gives a contradiction. From this, we assert the following for the case  $\xi \in \mathcal{Q}^\perp$ :

**Remark.** Let  $M$  is a Hopf hypersurface, that is,  $A\xi = \alpha\xi$  where  $\alpha = 2k$ , in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with GTW-Reeb parallel Ricci tensor,  $\widehat{\nabla}_\xi^{(k)} S = 0$ . If  $\xi \in \mathcal{Q}^\perp$ , then we only get the result that the trace  $h$  of the shape operator  $A$  is constant along the direction of  $\xi$ , that is,  $\xi h = 0$ .

From such a point of view, we now only focus our attention to the Ricci Reeb parallelism in GTW connection on the distribution  $\mathfrak{h} = \{X \in TM \mid X \perp \xi\}$ , as given by the proof of Theorem 2.

As mentioned above in the proof of Corollary 1, we see that  $\xi \in \mathcal{Q}$  or  $\xi \in \mathcal{Q}^\perp$ , because  $M$  is a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$  with  $\alpha = 2k$ . Moreover, if  $\xi \in \mathcal{Q}$ , then  $M$  must be a model space of Type (B).

Now, let us consider the case  $\xi \in \mathcal{Q}^\perp$ . Then by Suh [19] we have the following key lemma in the proof of Theorem 2.

**Lemma 3.1.** *Let  $M$  be a Hopf hypersurface, that is,  $A\xi = \alpha\xi$  where  $\alpha = 2k$ , in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . If  $M$  satisfies the following properties:*

- (i) *the Reeb vector field  $\xi$  belongs to the distribution  $\mathcal{Q}^\perp$ ,*
- (ii) *the Ricci tensor  $S$  is Reeb parallel with respect to both the Levi-Civita and GTW connections on  $\mathfrak{h}$ , that is,  $(\widehat{\nabla}_\xi^{(k)} S)X = 0$  and  $(\nabla_\xi S)X = 0$  for any tangent vector field  $W \in \mathfrak{h}$ ,*

*then  $M$  must be a model space of Type (A) or Type (B) in  $G_2(\mathbb{C}^{m+2})$ .*

*Proof.* As investigated above, from the assumption of  $\alpha = 2k$  and the equation (4.2) we have

$$(\xi h)AW = 0$$

for any tangent vector field  $W \in \mathfrak{h}$ .

From this, we see that the distribution  $\mathfrak{h}$  is totally geodesic, that is,  $AW = 0$  for any  $W \in \mathfrak{h}$ , if  $(\xi h) \neq 0$ . So, we can assert that  $M$  is a  $\mathcal{Q}^\perp$ -invariant hypersurface in  $G_2(\mathbb{C}^{m+2})$ , that is,  $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$ .

Next, we consider the case  $(\xi h) = 0$ . From (3.1) we get  $S\xi = (4m + h\alpha - \alpha^2)\xi$ . Differentiating this formula along the direction of  $\xi$  and using our assumptions,  $A\xi = \alpha\xi$ ,  $(\xi h) = (\xi\alpha) = 0$ , it follows that  $(\nabla_\xi S)\xi = 0$ . It implies that the Ricci tensor  $S$  becomes Reeb parallel. Then by virtue of the result given by Suh [19] we give a complete proof of our Lemma.  $\square$

As a consequence, we assert that if  $M$  is a Hopf hypersurface,  $\alpha = 2k$ , in  $G_2(\mathbb{C}^{m+2})$  satisfying two Ricci Reeb parallelism defined by  $(\nabla_\xi S)W = 0$  and  $(\widehat{\nabla}_\xi^{(k)} S)W = 0$  for any  $W \in \mathfrak{h}$ , then it must be either a real hypersurface of Type (A) or Type (B).

From now on, let us consider the converse problem. In other words, we now check whether the Ricci tensor  $S$  of model spaces  $M_A$  or  $M_B$  in  $G_2(\mathbb{C}^{m+2})$  satisfies the conditions in Theorem 2 or not.

By [2, Proposition 3] and the checking for a model space  $M_A$  given in the introduction and section 2, respectively, we see that  $M_A$  is a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$  with the GTW-Reeb parallel Ricci tensor on  $\mathfrak{h} \subset TM_A$ .

Now let us show that the Ricci tensor  $S$  of  $M_A$  is Reeb parallel in  $\nabla$  on  $\mathfrak{h}$ , that is,  $(\nabla_\xi S)W = 0$  for  $W \in \mathfrak{h} \subset TM_A$ . By virtue of [2, Proposition 3], the equation (3.3) can be written as

$$\begin{aligned} (\nabla_\xi S)Y &= h(\nabla_\xi A)Y - (\nabla_\xi A)AY - A(\nabla_\xi A)Y \\ (5.2) \quad &= h(\nabla_\xi A)Y - \tilde{\kappa}(\nabla_\xi A)Y - A(\nabla_\xi A)Y, \end{aligned}$$

where  $AY = \tilde{\kappa}Y$  for any  $W \in \mathfrak{h} \subset TM_A$ . Moreover, from the equation of Codazzi, we obtain

$$\begin{aligned} (\nabla_\xi A)Y &= (\nabla_Y A)\xi + \phi Y + \phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2 \\ (5.3) \quad &= \alpha\phi AY - A\phi AY + \phi Y + \phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2, \end{aligned}$$

since  $A\xi = \alpha\xi$  and  $h = \text{Tr}A = \alpha + 2\beta + (2m-2)\lambda$  where the eigenvalues  $\alpha, \beta, \lambda$  and  $\mu$  of  $M_A$  are constant. Since  $\mathfrak{h} = T_\beta \oplus T_\lambda \oplus T_\mu$ , let us check whether or not the Ricci tensor  $S$  of  $M_A$  satisfies the property of the Reeb-parallelism for each eigenspace.

**Case A-1:**  $Y \in T_\beta = \text{Span}\{\xi_2, \xi_3\}$

From (5.3), we obtain  $(\nabla_\xi A)\xi_2 = (\beta^2 - \alpha\beta - 2)\xi_3$ , which implies  $(\nabla_\xi A)\xi_2 = 0$  since  $\beta^2 - \alpha\beta - 2 = 0$ . So, we see that  $(\nabla_\xi S)\xi_2 = 0$  by (5.2). Similarly, if we put  $Y = \xi_3$  in (5.3), then  $(\nabla_\xi A)\xi_3 = -(\beta^2 - \alpha\beta - 2)\xi_2 = 0$ , because  $\alpha\beta = 2\cot^2(\sqrt{2}r) - 2$ . From this and (5.2), we see that  $(\nabla_\xi S)\xi_3 = 0$ .

**Case A-2:**  $Y \in T_\lambda = \{Y \perp \xi_1, \xi_2, \xi_3 \mid \phi Y = \phi_1 Y\}$

If  $Y \in T_\lambda$ , then  $\phi Y \in T_\lambda$ . From this, the equation (5.3) becomes  $(\nabla_\xi A)Y = (\alpha\lambda - \lambda^2 + 2)\phi Y$ . It follows  $(\nabla_\xi A)Y = 0$ , since  $\alpha\lambda = 2\tan^2(\sqrt{2}r) - 2$ . Hence we see that the Ricci tensor  $S$  of  $M_A$  becomes Reeb parallel on  $T_\lambda$ , that is,  $(\nabla_\xi S)Y = 0$  for any  $Y \in T_\lambda$ .

**Case A-3:**  $Y \in T_\mu = \{Y \perp \xi_1, \xi_2, \xi_3 \mid \phi Y = -\phi_1 Y\}$

Since  $Y \in T_\mu$ , then  $\phi Y = -\phi_1 Y$  and  $\mu = 0$ . From these, the equation (5.3) becomes  $(\nabla_\xi A)Y = 0$  for  $Y \in T_\mu$ . Hence it implies that the Ricci tensor of  $M_A$  is Reeb parallel on  $T_\mu$ , that is,  $(\nabla_\xi S)Y = 0$  for any  $Y \in T_\mu$ .

Summing up three cases above,  $M_A$  have Reeb parallel Ricci tensor in the Levi-Civita connection  $\nabla$  on the distribution  $\mathfrak{h}$ .

On the other hand, let us check whether  $M_B$  satisfies our conditions,  $\nabla_\xi S = 0$  and  $\widehat{\nabla}_\xi^{(k)} S = 0$  on  $\mathfrak{h} \subset TM_B$ . Suppose that the Ricci tensor  $S$  of  $M_B$  is Reeb parallel,  $(\nabla_\xi S)X = 0$  for  $X \in \mathfrak{h}$ . From (3.3) and (4.9) we obtain

$$(\nabla_\xi S)X = \begin{cases} (-4\alpha + h\alpha\beta - \alpha\beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ -4(\alpha + h - \beta)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (h - \beta)(\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ (h - \beta)(\alpha\mu + 2)\phi X & \text{if } X \in T_\mu. \end{cases}$$

Since the Ricci tensor  $S$  is Reeb parallel on the eigenspace  $T_\lambda$ , we have  $(h - \beta)(\alpha\lambda + 2) = 0$ . It implies that

$$(5.4) \quad (h - \beta) = 0,$$

because  $(\alpha\lambda + 2) \neq 0$ . On the other hand, for  $T_\gamma$  we get  $(\alpha + h - \beta) = 0$ , which means  $\alpha = 0$  from (5.4). It makes a contraction. Thus we assert that there does not exist  $M_B$  satisfying the conditions in Theorem 2.

With such assertions we give a complete proof of Theorem 2 in the introduction.  $\square$

#### 4. PROOFS OF THEOREM 3 AND COROLLARY 2

First we want to give a proof of Theorem 3. Among the conditions in Theorem 2, we focus our attentions to the assumptions related to the Reeb parallelism of Ricci tensor  $S$ . Actually, we consider that on  $\mathfrak{h}$  two covariant derivatives of  $S$  in Levi-Civita and GTW connections are equal to zero, that is,  $(\nabla_\xi S)W = 0 = (\widehat{\nabla}_\xi^{(k)} S)W$  for any tangent vector field  $W \in \mathfrak{h} = \{X \in TM \mid X \perp \xi\}$ . So, in this section, we will

consider the following condition related to the Reeb parallelism of Ricci tensor  $S$ .

$$(C-2) \quad (\nabla_\xi S)X = (\widehat{\nabla}_\xi^{(k)} S)X$$

for any tangent vector field  $X$  on  $M$ . By virtue of the equation (3.4), the condition (C-2) is equivalent to the  $S\phi = \phi S$ . On the other hand, Suh proved in [17] that a Hopf hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , with commuting Ricci tensor is locally congruent a tube of radius  $r$  over a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ . Then we conclude that a Hopf hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , satisfying the condition (C-2) if and only if  $M$  is of Type (A), which gives us a complete proof of Theorem 3.

By Theorem 3, if a real hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$  satisfies  $\nabla S = \widehat{\nabla}^{(k)} S$ , then naturally (C-2) holds on  $M$ . So  $M$  is of Type (A). Now let us check whether a model space  $M_A$  of Type (A) satisfies our condition

$$(C-3) \quad (\widehat{\nabla}_X^{(k)} S)Y = (\nabla_X S)Y$$

for any tangent vector fields  $X, Y \in T_x M_A$ ,  $x \in M_A$ . In order to do this, we assume that the Ricci tensor  $S$  of  $M_A$  satisfies (C-3). That is, we have

$$(6.1) \quad \begin{aligned} 0 &= (\widehat{\nabla}_X^{(k)} S)Y - (\nabla_X S)Y \\ &= g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\ &\quad - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y \end{aligned}$$

for any  $X, Y \in T_x M_A$ .

Since  $T_x M_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\gamma$ , the equation (6.1) holds for  $X \in T_\beta$  and  $Y \in T_\alpha$ . For the sake of convenience we put  $X = \xi_2 \in T_\beta$  and  $Y = \xi \in T_\alpha$ . Since  $S\xi = \delta\xi$  and  $S\xi_3 = \sigma\xi_3$  where  $\delta = (4m + h\alpha - \alpha^2)$  and  $\sigma = (4m + 6 + h\beta - \beta^2)$ , the equation (6.1) reduces to  $\beta(\delta - \sigma)\xi_3 = 0$ . By [2, Proposition 3], since the principal curvature  $\beta = \sqrt{2} \cot(\sqrt{2}r)$  for  $r \in (0, \pi/\sqrt{8})$  is non-zero, it follows  $(\delta - \sigma) = 0$ . In other words, by [2, Proposition 3] we obtain

$$\begin{aligned} -(\delta - \sigma) &= 6 - \alpha\beta + \beta^2 + (2m - 2)\beta\lambda - (2m - 2)\alpha\lambda \\ &= 8 - 4(m - 1)\tan^2(\sqrt{2}r), \end{aligned}$$

which gives us

$$(6.2) \quad \tan^2(\sqrt{2}r) = \frac{2}{m - 1}.$$

In addition, since (6.1) holds for  $X \in T_\lambda$  and  $Y = \xi$ , we obtain

$$0 = (\widehat{\nabla}_X^{(k)} S)\xi - (\nabla_X S)\xi = \lambda(\tau - \delta)\phi X,$$

where in the second equality we have used  $\phi X \in T_\lambda$  and  $SX = (4m + 6 + h\lambda - \lambda^2)X = \tau X$  for any  $X \in T_\lambda$ . Because  $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$  where  $r \in (0, \pi/\sqrt{8})$  is non-zero, we have also

$$\tau - \delta = 0.$$

By straightforward calculation it is

$$\begin{aligned} \tau - \delta &= 6 + h\lambda - \lambda^2 - h\alpha + \alpha^2 \\ &= 4m - 4\cot^2(\sqrt{2}r) = 0. \end{aligned}$$

From (6.2), it becomes  $2m + 2 = 0$ , which gives us a contradiction. Accordingly, it completes our Corollary 2 given in the introduction.  $\square$

## REFERENCES

- [1] D. V. Alekseevskii, Compact quaternion spaces, *Func. Anal. Appl.*, **2** (1966), 106-114.
- [2] J. Berndt and Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians, *Monatsh. Math.*, **127** (1999), 1-14.
- [3] J.T. Cho, CR structures on real hypersurfaces of a complex space form, *Publ. Math. Debrecen*, **54** (1999), 473-487.
- [4] J.T. Cho, Levi parallel hypersurfaces in a complex space form, *Tsukuba J. Math.*, **30**(2006), 329-344.
- [5] I. Jeong, M. Kimura, H. Lee and Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster Reeb parallel shape operator, *Monatsh. Math.*, **171**, (2013), 357-376
- [6] I. Jeong, H. Lee and Y.J. Suh, Levi-Civita and generalized Tanaka-Webster covariant derivatives for real hypersurfaces in complex two-plane Grassmannians (submitted).
- [7] I. Jeong, Machado, Carlos J. D. Pérez and Y. J. Suh, Real hypersurfaces in complex two-plane Grassmannians with  $\mathcal{D}^\perp$ -parallel structure Jacobi operator, *Inter. J. Math.*, **22** (2011), no. 5, 655-673.
- [8] H. Lee, Y.S. Choi, and C. Woo, Hopf hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator, *Bull. Malaysian Math. Soc.* (2014) (in press).
- [9] H. Lee and Y.J. Suh, Real hypersurfaces of type  $B$  in complex two-plane Grassmannians related to the Reeb vector, *Bull. Korean Math. Soc.*, **47** (2010), no. 3, 551-561.
- [10] C.J.G. Machado and J.D. Pérez, Real hypersurfaces in complex two-plane Grassmannians some of whose Jacobi operators are  $\xi$ -invariant, *Internat. J. Math.*, **23** (2010), 1250002 (12pages).
- [11] C.J.G. Machado and J.D. Pérez, On the structure vector field of a real hypersurfaces in complex two-plane Grassmannians, *Cent. Eur. J. Math.*, **10** (2010), 451-455.
- [12] J.D. Pérez and F.G. Santos, Real hypersurfaces in complex projective space whose structure Jacobi operator is cyclic-Ryan parallel, *Kyungpook Math. J.*, **49** (2009), 211-219.
- [13] J.D. Pérez and Y.J. Suh, The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians, *J. Korean Math. Soc.*, **44** (2007), 211-235.
- [14] J.D. Pérez and Y.J. Suh, On the Ricci tensor of a real hypersurface in complex two-plane Grassmannians (submitted).
- [15] J.D. Pérez and Y.J. Suh, Generalized Tanaka-Webster and covariant derivatives on a real hypersurface in a complex projective space (submitted).
- [16] J.D. Pérez and Y.J. Suh, and Y. Watanabe. Generalized Einstein real hypersurfaces in complex two-plane Grassmannians. *J. Geom. Phys.*, **60** (2010), 1806-1818.
- [17] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor, *J. Geom. Phys.*, **60** (2010), 1792-1805.
- [18] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor, *Proc. Royal Soc. Edinb. A.*, **142** (2012), 1309-1324.
- [19] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with Reeb parallel Ricci tensor, *J. Geome. Phys.*, **64** (2013), 1-11.

HYUNJIN LEE

THE CENTER FOR GEOMETRY AND ITS APPLICATIONS,  
POHANG UNIVERSITY OF SCIENCE & TECHNOLOGY,  
POHANG 790-784, REPUBLIC OF KOREA  
*E-mail address:* lhjibis@hanmail.net

YOUNG JIN SUH AND CHANGHWA WOO  
DEPARTMENT OF MATHEMATICS,  
KYUNGPOOK NATIONAL UNIVERSITY,  
DAEGU 702-701, REPUBLIC OF KOREA  
*E-mail address:* yjsuh@knu.ac.kr  
*E-mail address:* legalgwch@knu.ac.kr